# The Complexity of the Separable Hamiltonian Problem

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#### Abstract

In this paper, we study variants of the canonical LOCAL HAMILTONIAN problem where, in addition, the witness is promised to be separable. We define two variants of the LOCAL HAMILtonian problem. The input for the separable local hamiltonian problem is the same as the LOCAL HAMILTONIAN problem, i.e. a local Hamiltonian and two energies  $a$  and  $b$ , but the question is somewhat different: the answer is yes if there is a *separable* quantum state with energy at most a, and the answer is no if all separable quantum states have energy at least b. The separable sparse hamiltonian problem is defined similarly, but the Hamiltonian is not necessarily local, but rather sparse. We show that the SEPARABLE SPARSE HAMILTONIAN problem is QMA(2)-complete, while separable local hamiltonian is in QMA. This should be compared to the local hamiltonian problem, and the sparse hamiltonian problem which are both QMA-complete. To the best of our knowledge, SEPARABLE SPARSE HAMILTONIAN is the first non-trivial problem shown to be QMA(2)-complete.

## 1 Introduction and Results

### 1.1 Introduction

The class QMA is the the quantum analogue of the class NP (or more precisely, MA). The class was first studied by Kitaev [\[KSV02\]](#page-13-0), and has been in focus since: see [\[AN02\]](#page-12-0) for a survey, and [\[Osb11\]](#page-13-1) for a more recent physics-motivated review. First, Watrous showed that GROUP-NON-MEMBERSHIP is in QMA [\[Wat00\]](#page-13-2) (this problem is not known to be in MA). Then, after a series of works, Kempe, Kitaev and Regev showed that the 2-LOCAL HAMILTONIAN problem is QMA-complete [\[KKR04\]](#page-13-3). Mariott and Watrous also proved a strong amplification result of QMA [\[MW05\]](#page-13-4). More recently, Aharonov et al. tried to extend the celebrated PCP theorem to the quantum case [\[AALV09\]](#page-12-1). QMA, as the quantum equivalent of NP, is one of the most studied classes in quantum complexity.

One of the striking results in proof systems is that sometimes, limiting the prover can increase the power of the proof system. For example  $IP = PSPACE$  [\[LFKN92,](#page-13-5) [Sha92\]](#page-13-6), while MIP = NEXP [\[BFL91\]](#page-12-2). This means that two classical provers can prove more languages to a verifier if it is guaranteed that the provers cannot communicate with each other. However, these classical examples require interaction between the prover and the verifier. The class  $\mathsf{QMA}(k)$ , introduced by Kobayashi

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et al. [\[KMY03\]](#page-13-7), deals with quantum non-interactive proofs and limits the prover to send k nonentangled proofs, or equivalently k-unentangled provers that cannot communicate with each other. The question whether  $QMA(k) = QMA(2)$  was answered in the affirmative by Harrow and Monta-naro [\[HM10\]](#page-13-8). The question whether  $QMA(2) \subseteq QMA$  is still open. Note that in the classical case,  $MA(k) = MA(2) = MA$ .

To show the power of unentangled quantum proofs, Blier and Tapp [\[BT09\]](#page-12-3) first presented a  $QMA(2)$  protocol for an NP-complete problem with two quantum witnesses of size  $O(log(n))$ . The drawback of this protocol is that the soundness parameter is somewhat disappointing  $(1 - \Omega(1/n^6))$ . This was first improved by Beigi [\[Bei10\]](#page-12-4) who showed that the soundness can be reduced to 1 −  $1/n^{3+\varepsilon}$  for any  $\varepsilon > 0$ . Very recently, Le Gall improved this soundness to  $1 - \Omega(\frac{1}{n \log(n)})$  [\[LNN11\]](#page-13-9). Aaronson *et al.* showed that there exists a short proof for SAT in  $\mathsf{QMA}(\tilde{O}(\sqrt{n}))$  [\[ABD](#page-12-5)<sup>+</sup>08], where each unentangled witness has logarithmic size, but where the soundness can be exponentially small. In [\[HM10\]](#page-13-8) it was shown that SAT  $\in QMA(2)$ , where the size of each proof is  $O(\sqrt{n})$ . These results tend to show that quantum unentangled proofs are very powerful, since they can solve NP-complete problems in a seemingly more efficient way than in QMA. Liu *et al.* have shown that PURE  $N$  – REPRESENTABILITY, an important problem in quantum chemistry, is in  $\mathsf{QMA}(2)[\text{LCVO7}]$ . This problem is not known to be in QMA.

On the other hand, Brandão et al. [\[BCY11\]](#page-12-6) showed that if the verifier is restricted to performing a Bell measurement, then, the resulting class BELL-QMA(2) is equal to QMA. Trying to understand the relationship between QMA and QMA(2) is a fundamental open problem from the point of view of quantum complexity as well as for the understanding of the power of quantum unentangled proofs.

### 1.2 Contribution

In this paper, we study the relationship between QMA and QMA(2) from a different perspective. We study the LOCAL HAMILTONIAN problem with unentangled witnesses. The  $k$  –LOCAL HAMILTONIAN (see Def. [6\)](#page-4-0) problem is the quantum analog of  $MAX-k-SAT$ , and is the canonical QMA-complete problem. The first proof that k – LOCAL HAMILTONIAN is QMA-complete is by Kitaev. Our first result is to extend this construction to separable witnesses in order to find a complete problem for QMA(2). The main ingredient in showing that the  $k -$  LOCAL HAMILTONIAN problem is QMA-complete, is Kitaev's Hamiltonian, a Hamiltonian which penalizes states that are not history states. History states are states of the form  $|\eta_{\psi}\rangle \equiv \frac{1}{\sqrt{T}}$  $\frac{1}{T+1}\sum_{t=0}^{T}|t\rangle \otimes |\psi_t\rangle$ , where  $|\psi_t\rangle$  is the state at the t-th step of the verification process when starting with  $|\psi\rangle$  and the the m ancilla qubits in 0 state, i.e.  $|\psi_t\rangle = U_t U_{t-1} \dots U_0(|0^m\rangle \otimes |\psi\rangle)$ , and  $U_i$  is the *i*-th gate used in the QMA verification circuit, and we set as a convention  $U_0 = I$ .

It is natural to try to adapt this argument to a QMA(2) verification circuit by constructing a separable local hamiltonian problem: the input for the separable local hamiltonian problem is the same as the local hamiltonian problem, i.e. a collection of local Hamiltonians  ${H_1, \ldots, H_m}$ , the answer is yes if there is a *separable* quantum state with energy at most a, and the answer is no if all separable quantum states have energy at least b for some energies  $a < b$ . Yet, there is a flaw in this argument: even if  $|\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$ , the history state  $|\eta_{\psi}\rangle$  might not be separable. This is caused by two reasons:(i) even though  $|\psi_0\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$  is a tensor product, for  $t > 0$ ,  $|\psi_t\rangle$  can be entangled, and (ii) even if  $|\psi_t\rangle$  is not entangled, the fact that  $|\eta_{\psi}\rangle$  is a superposition over all time steps creates entanglement, as long as both parts of the proof change during the computation.

In order to resolve the entanglement issue in  $|\eta_{\psi}\rangle$ , we use the construction of Harrow and Montanaro [\[HM10\]](#page-13-8). They show that every  $\mathsf{QMA}(k)$  verification circuit can be transformed into a QMA(2) circuit with the following structure: The first and second witnesses(which are promised to be non-entangled) have the same length, where each witness contains  $r$  registers, where each register size, in the first and second witnesses, is the same. The first  $r$  steps of the verification procedure are swap-tests between the i-th register of the first and second witnesses, and from that point, the verification circuit acts non-trivially only on the first witness. In a yes instance, there exists a non-entangled proof, where  $|\chi_A\rangle = |\chi_B\rangle = |\chi_1\rangle \otimes |\chi_2\rangle \otimes \ldots \otimes |\chi_r\rangle$ . Notice that  $C-SWAP(|+\rangle\otimes |\phi\rangle\otimes |\phi\rangle=|+\rangle\otimes |\phi\rangle\otimes |\phi\rangle$ , therefore, applying the swap-tests to the above witnesses does not change the state. Since there are no other operations on the second witness, the second witness remains fixed during the entire verification process. If we treat the clock, ancilla qubits and the first witness as the  $A$  system, and the second witness as the  $B$  system, we get that the history state  $|\eta\rangle$  is indeed separable with respect to this division. This is only true if the controlled swap operation is applied on all the qubits in the *i*-th register of the first and second witnesses. This will make the propagation terms<sup>[1](#page-2-0)</sup> in Kitaev's Hamiltonian non-local. But, on the other hand, a controlled swap operation on arbitrary number of qubits is always sparse: each row has one non-zero entry. This makes each propagation term sparse.

Given a sparse Hamiltonian H, the unitary  $U = \exp(-iHt)$  can be implemented efficiently, which eventually leads to SEPARABLE SPARSE HAMILTONIAN  $\in QMA(2)$ . Together with the idea above, it can be shown that:

#### <span id="page-2-1"></span>**Theorem 1.** SEPARABLE SPARSE HAMILTONIAN is  $QMA(2)$ -complete.

The only reason why, this construction does not lead to a SEPARABLE LOCAL HAMILTONIAN instance, is that the controlled swap gate must be performed in one step; otherwise,  $|\eta\rangle$  would become entangled. At first glance, this might seem as a technicality, but we, surprisingly, show that:

#### <span id="page-2-2"></span>Theorem 2. SEPARABLE LOCAL HAMILTONIAN is QMA-complete.

Since the SEPARABLE LOCAL HAMILTONIAN problem is at least as hard as the LOCAL HAMILTOnian problem, and local hamiltonian is QMA-complete, therefore separable local hamil-TONIAN is QMA-hard. To show that SEPARABLE LOCAL HAMILTONIAN  $\in QMA$ , we use the CONsistency of local density matrices problem [\[Liu06\]](#page-13-11) as a subroutine. Informally, the consistency of local density matrices promise problem asks the following question: given a collection of local density matrices  $\rho_i$  over a constant set of qubits  $C_i$ , is there a quantum state  $\rho$ such that for each *i*, the reduced density matrix of  $\rho$  over the qubits  $C_i$  is equal to  $\rho_i$ ? Liu showed that this problem is QMA-complete.

To show that separable local hamiltonian is QMA-complete, we do as follows. Assume that there exists a state  $\sigma = \sigma_A \otimes \sigma_B$  of total length 2n, with energy below the threshold a. Let  $A, B$ the two spaces of qubits considered, each of size n. The energy is  $tr(H(\sigma_A \otimes \sigma_B))$  where  $H = \sum_i H_i$ . Let  $C_i$  the subset of qubits each  $H_i$  act on. We have  $\text{tr}(H(\sigma_A \otimes \sigma_B)) = \sum_{i=1}^m \text{tr}(H_i \sigma^{C_i})$ , where  $\sigma^{C_i}$ corresponds to the reduced state of  $\sigma$  on the qubits of  $C_i$ . Again, we can decompose  $\sigma^{C_i}$  into the A part and the B part. We can write  $\sigma_{ci}^{C_i} = \sigma^{A_i} \otimes \sigma^{B_i}$ . This is because the state  $\sigma$  is a product state between A and B, hence, the state  $\sigma^{C_i}$  is also a product state between A and B.

<span id="page-2-0"></span><sup>&</sup>lt;sup>1</sup>See Eq.  $(1)$  for the definition.

The proof will consist of a classical part: the classical description of the reduced density matrices  $\sigma^{A_i}, \sigma^{B_i}$ . This information is sufficient to calculate the energy classically, using  $\text{tr}(H(\sigma_A \otimes \sigma_B))$  =  $\sum_{i=1}^m \text{tr}(H_i(\sigma^{A_i} \otimes \sigma^{B_i}))$ . The proof also consists of a quantum part: the prover tries to convince the verifier that there exists a quantum mixed state  $\rho_A$  and similarly for  $\rho_B$  that are consistent with the reduced density matrices  $\sigma^{A_i}$  and  $\sigma^{B_i}$ . Since CONSISTENCY OF LOCAL DENSITY MATRICES is known to be in QMA, the prover can convince the verifier if there exists such a state, but cannot fool the verifier if there is no such state.

Discussion In the case, where we do not consider separable witnesses, the two problems LOCAL hamiltonian and sparse hamiltonian are natural QMA-complete problems. Thus, in this setting, considering sparse Hamiltonians instead of local Hamiltonians does not increase the power of the verifier.

When we consider separable witnesses, things are different. SEPARABLE LOCAL HAMILTONIAN and SEPARABLE SPARSE HAMILTONIAN seem to be natural QMA(2)-complete problems. With Theo-rem [1,](#page-2-1) we show that SEPARABLE SPARSE HAMILTONIAN is indeed QMA(2)-complete by adapting Kitaev's completeness and using the new construction from Harrow and Montanaro [\[HM10\]](#page-13-8). However, we were not able to remove this sparseness condition to show that SEPARABLE LOCAL HAMILTONIAN is also QMA(2)-complete.

On the other hand, we show that separable local hamiltonian is QMA-complete. We find this surprising because separable local hamiltonian was a natural candidate for a QMA(2)-complete problem. This also means that when considering separable witnesses, the sparse condition for Hamiltonians is crucial or conversely that separable witnesses do not help a verifier when his accepting procedure is a sum of local Hamiltonians. This is in sharp contrast with the general case where separable witnesses seem to help the verifier significantly. While we do not have a clear separation between QMA and QMA(2), we know that  $QMA \subseteq PP \subseteq PSPACE$  (first unpublished proof by Kitaev and Watrous then simplified in [\[MW05\]](#page-13-4)) while we only know that  $\mathsf{QMA} \subseteq \mathsf{QMA}(2) \subseteq \mathsf{NEXP}$  [\[KMY03\]](#page-13-7).

Our results characterize rather tightly the difference between QMA and QMA(2). We hope that this will lead to a better understanding of the relationship between these classes.

Structure of the paper: Section [2](#page-3-0) contains the preliminaries and definitions. In Section [3,](#page-5-0) we show that SEPARABLE SPARSE HAMILTONIAN is  $QMA(2)$ -complete (Theorem [1\)](#page-2-1). In Section [4,](#page-10-0) we show that SEPARABLE LOCAL HAMILTONIAN is QMA-complete (Theorem [2\)](#page-2-2).

## <span id="page-3-0"></span>2 Preliminaries and Definitions

**Definition 3.** A promise problem  $L = \{L_{yes}, L_{no}\}\$ is in  $\mathsf{QMA}_{s,c}(k)$  if there exists a uniformly generated polynomial time quantum algorithm A and computable polynomially bounded functions  $f_1, \ldots, f_k$  such that for all input  $x \in \{0,1\}^n$ :

- 1. **Completeness:** if  $x \in L_{yes}$  there exist k witnesses  $|\psi_1\rangle, \ldots, |\psi_k\rangle$ , where each witness  $|\psi_i\rangle$ consists of  $f_i(n)$  qubits such that A accepts  $|x\rangle \otimes |\psi_1\rangle \otimes \ldots \otimes |\psi_k\rangle$  with probability at least c.
- 2. Soundness: if  $x \in L_{no}$  then for all k witnesses  $|\psi_1\rangle, \ldots, |\psi_k\rangle$ , where each witness  $|\psi_i\rangle$  consists of  $f_i(n)$  qubits the probability that A accepts  $|x\rangle \otimes |\psi_1\rangle \otimes \ldots \otimes |\psi_k\rangle$  is at most s.

We define  $\mathsf{QMA}_{\frac{1}{3},\frac{2}{3}}(k) = \mathsf{QMA}(k)$ , and  $\mathsf{QMA} = \mathsf{QMA}(1)$ .

<span id="page-4-4"></span>**Theorem 4** ([\[HM10\]](#page-13-8)). If  $c-s \geq 1/poly(n)$ ,  $k = poly(n)$ , and  $p(n)$  is an arbitrary polynomial, then  $QMA_{s,c}(k) = QMA_{2^{-p(n)},1-2^{-p(n)}}(2).$ 

Furthermore, it can be assumed w.l.o.g. that the  $QMA(2)$  protocol has the following structure: The two witnesses have exactly the same size, where both of them consist of r registers of sizes  $s_1, \ldots, s_r$ . The verification process consists of applying the product test (see Def. [5\)](#page-4-1). If the product test fails, Arthur rejects. Otherwise, Arthur runs a polynomial quantum algorithm A on the first proof, and outputs the result. In a yes instance, the two Merlins can send identical states, which are tensor product between the r registers:  $|\psi_1\rangle = |\psi_2\rangle = |\chi_1\rangle \otimes \ldots \otimes |\chi_r\rangle$ .

<span id="page-4-1"></span>Definition 5 (Product Test[\[HM10\]](#page-13-8)). The input consists of two states, where each state has r registers of size  $s_1, \ldots, s_r$ . Preform the swap test on each of the r pairs. Accept if all of the swaptests pass, otherwise reject.

<span id="page-4-0"></span>**Definition 6** (k-LOCAL HAMILTONIAN problem). Input: a set of hermitian matrices  $H_1, \ldots, H_m$ , where each matrix operates on a set of at most k out of the n qubits, and  $I \succeq H_i \succeq 0$  (i.e. both  $H_i$ and  $I - H_i$  are positive semi definite), and two real number a and b such that  $b - a > poly(1/n)$ . We define the Hamiltonian, with a slight abuse of notation<sup>[2](#page-4-2)</sup>,  $H = \sum_{i=1}^{m} H_i$ . Output: Output YES if there exists a state  $|\psi\rangle$  such that  $\langle \psi | H | \psi \rangle \le a$ , and no if for every state  $|\psi\rangle$ ,  $\langle \psi | H | \psi \rangle \ge b$ .

Definition 7 (Simulatable Hamiltonian[\[ATS03\]](#page-12-7)). We say a Hamiltonian H on n qubits is simulatable if for every  $t > 0$  and every accuracy  $\alpha > 0$ , the unitary transformation  $U = \exp(-iHt)$  can be approximated to within  $\alpha$  accuracy by a quantum circuit of size  $poly(n, t, \frac{1}{\alpha})$ .

<span id="page-4-3"></span>**Definition 8** (SEPARABLE k-LOCAL HAMILTONIAN problem). The input is the same as the input for the k-LOCAL HAMILTONIAN problem together with a partition of the qubits to disjoint sets A and B. The answer is  $\text{Yes } i \in \mathbb{F} \Rightarrow |\chi_A\rangle \otimes |\chi_B\rangle \text{ s.t. } \langle \psi | H | \psi \rangle \leq a \text{ and the answer is no if } \langle \psi | H | \psi \rangle \geq b$ for all tensor product states  $|\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$ .

Remark: The above definition can be formulated using mixed states in the two following ways, with mixed product states and mixed separable states. It can be verified that indeed these definitions are equivalent.

<span id="page-4-5"></span>**Definition 9** (SEPARABLE k-LOCAL HAMILTONIAN problem - alternative definition 1). The input is the same as in Def. [8.](#page-4-3) The answer is  $\gamma$ ES if there exists a product mixed state  $\rho = \rho_A \otimes \rho_B$  s.t.  $tr(H\rho) \leq a$  and the answer is NO if  $tr(H\rho) \geq b$  for all product mixed states  $\rho = \rho_A \otimes \rho_B$ .

Definition 10 (SEPARABLE k-LOCAL HAMILTONIAN problem - alternative definition 2). The input is the same as in Def. [8.](#page-4-3) The answer is YES if there exists a separable mixed state  $\rho = \sum_i p_i ( \rho_i^A \otimes$  $\rho^B_{i,j}$  s.t. tr $(H\rho) \le a$  and the answer is NO if tr $(H\rho) \ge b$  for all separable mixed states  $\rho = \sum_i p_i(\rho^A_i \otimes$  $\rho_i^B$ ).

We now define the SEPARABLE SPARSE HAMILTONIAN problem.

**Definition 11** (SEPARABLE SPARSE HAMILTONIAN). An operator A over n qubits is row-sparse if each row in A has at most  $poly(n)$  non-zero entries, and there exists an efficient classical algorithm that, given i, outputs a list  $(j, A_{i,j})$  running over all non zero elements of  $A_{i,j}$ . The SEPARABLE SPARSE HAMILTONIAN problem is the same as SEPARABLE  $k$ -LOCAL HAMILTONIAN except each term in the input Hamiltonian is row-sparse instead of k-local.

<span id="page-4-2"></span><sup>&</sup>lt;sup>2</sup>Each matrix  $H_i$  operates on some set of qubits, and the summation is over their extension to the entire Hilbert space of the *n* qubits.

Finally, we define the CONSISTENCY OF LOCAL DENSITY MATRICES problem which we will use to show that the separable k-local hamiltonian problem is QMA-complete.

<span id="page-5-3"></span>**Definition 12** (CONSISTENCY OF LOCAL DENSITY MATRICES [\[Liu06\]](#page-13-11)). We are given a collection of local density matrices  $\rho_1, \ldots, \rho_m$ , where each  $\rho_i$  is a density matrix over qubits  $C_i \subset \{1, \ldots, n\}$ , and  $|C_i| \leq k$  for some constant k. Each matrix entry is specified by poly $(n)$  bits of precision. In addition, we are given a real number  $\beta \geq 1/poly(n)$  specified with poly(n) bits of precision. The problem is to distinguish between the following two cases:

- 1. There exists an n qubits mixed state  $\sigma$  such that for all i such that  $\text{Tr}_{\{1,\ldots,n\}\backslash C_i}(\sigma) = \rho_i$ . In this case, output yes.
- 2. For all n qubits mixed states  $\sigma$ , there exists some i such that  $||\text{Tr}_{\{1,\ldots,n\}\setminus C_i}(\sigma) \rho_i||_1 \geq \beta$ . In this case output no.

### <span id="page-5-0"></span>3 Proof that separable sparse hamiltonian is QMA(2)-complete

### 3.1 SEPARABLE SPARSE HAMILTONIAN  $\in QMA(2)$

The construction has the same structure as the proof that LOCAL HAMILTONIAN  $\in QMA$  in  $|KSV02|$ , and uses phase estimation as a subroutine to achieve that [\[KSV02,](#page-13-0) [NC00\]](#page-13-12). We consider row sparse Hamiltonians  $\{H_j\}_{1\leq j\leq m}$  and  $H=\sum_{j=1}^m H_j$ . For each j, we construct a quantum algorithm  $Q_j$ such that

$$
|\Pr(Q_j \text{ accepts } |\psi\rangle) - (1 - \langle \psi | H_j | \psi \rangle)| \le \varepsilon,
$$

where we choose  $\varepsilon = \frac{b-a}{3}$ , and the running time of  $Q_j$  is polynomial in n. Let Q be the algorithm where we pick  $1 \leq j \leq m$  at random, and run  $Q_j$ .

$$
|\Pr(Q \text{ accepts } |\psi\rangle) - (1 - \frac{1}{m} \langle \psi | H | \psi \rangle)| \le \varepsilon.
$$

Therefore, in a yes instance there exists a state  $|\psi\rangle = |\chi_A\rangle \otimes |\chi_B\rangle$  which is accepted with probability at least  $c = 1 - \frac{a}{m} - \varepsilon$ , whereas in a NO instance, the probability of acceptance for every tensor product state is at most  $s = 1 - \frac{b}{m} + \varepsilon$ . Therefore the problem is in  $\mathsf{QMA}_{c,s}(2)$ , which is equal to  $QMA(2)$  by Thm. [4.](#page-4-4)

All that is left to show how to implement  $Q_i$ . Aharonov and Ta-Shma have shown:

<span id="page-5-2"></span>**Lemma 13** (The sparse Hamiltonian lemma  $[ATS03]$ ). If H is row-sparse, and  $||H|| \leq poly(n)$  then H is simulatable.

<span id="page-5-1"></span>**Theorem 14** (Phase Estimation  $[NC00]$ ). Let V be a unitary which can be implemented by a quantum circuit with d gates, which has eigenstates  $\{|u_j\rangle\}_{1\leq j\leq N}$ , and eigenvalues  $\{e^{i\phi_j}\}_{1\leq j\leq N}$ . Given a state  $|\phi\rangle = \sum_{i=1}^{N} \sqrt{p_i} |u_i\rangle$ , an error parameter  $\epsilon$  and a precision parameter  $\delta$ , the phase estimation procedure outputs with probability at least  $p_i(1-\varepsilon)$  a number which is  $\delta$  close to  $\phi_i$ .

Let  $t = \log(\delta) + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$ . The phase estimation procedure can be implemented by a quantum circuit with  $O(t^2 + d^{2^t})$  gates.

We can now show how to implement  $Q_i$ :

1. Start with a state  $|\psi\rangle = \sum_{i=1}^{N} \sqrt{p_i} |u_i\rangle$ , where  $|u_i\rangle$  is an eigenstate of  $H_j$  with eigenvalue  $\phi_i$ .

- 2. Apply phase estimation with the unitary  $U = \exp(iH_i)$  with probability for an error and precision  $\frac{b-a}{6}$ . Denote by  $\tilde{\phi}$  the output of the phase estimation.
- 3. Reject with probability  $\phi$ .

Using Thm. [14,](#page-5-1) we can get both a lower and an upper bound on the acceptance probability:

$$
(1 - \frac{b-a}{6})\sum_{i=1}^{N} p_i(\phi_i - \frac{b-a}{6}) \le Pr(Q_j \text{ rejects } |\psi\rangle) \le \frac{b-a}{6} + (1 - \frac{b-a}{6})\sum_{i=1}^{N} p_i(\phi_i + \frac{b-a}{6}).
$$

Since  $\sum_{i=1}^{N} p_i \phi_i = \langle \psi | H_j | \psi \rangle$ , and  $I \succeq H_i \succeq 0$  we get:

$$
\langle \psi | H_j | \psi \rangle - \frac{b-a}{3} \le Pr(Q_j \text{ rejects } |\psi \rangle) \le \langle \psi | H_j | \psi \rangle + \frac{b-a}{3},
$$

which was the requirement for  $Q_i$ .

Unfortunately, By Lemma [13,](#page-5-2) we can only approximate  $U = \exp(iH_i)$  which is needed in step 2. A polynomial approximation, which can be achieved in polynomial time, is good enough for our needs, for similar reasons as the analysis done in [\[ATS03\]](#page-12-7) and [\[WZ06,](#page-13-13) Section 4.1].

## 3.2 separable sparse hamiltonian is QMA(2)-hard

Consider a promise problem  $L = \{L_{yes}, L_{no}\}\$  which is in  $\mathsf{QMA}_{s,c}(2)$  with  $c = 1 - \frac{C}{512(T+1)^4}$  and  $s = \frac{1}{T+1}$ . C is a universal constant that will be specified later. For such s and c, we have  $\mathsf{QMA}_{s,c}(2)$ QMA(2) by Theorem [4.](#page-4-4) Our goal is to reduce this problem to the separable sparse hamiltonian problem.

Pick an instance x of L and let A the associated verifying procedure. We will omit the dependence in  $x$  and write the verifying procedure as a unitary  $U$  taking as input the two quantum witnesses. We can assume w.l.o.g. that this verification procedure has the structure described in Thm. [4.](#page-4-4) We decompose the verifying procedure into T unitaries  $U = U_1, \ldots, U_T$  each acting on a 2 qubits. This means that after t steps of the verifying procedure, the unitary applied is  $U_tU_{t-1}\cdots U_0$ , where we add the convention that  $U_0 = I$ .

We apply Kitaev's construction (See [\[KSV02,](#page-13-0) Sec. 14.4.1] for the detailed definition) of the circuit, and get a Hamiltonian of the form<sup>[3](#page-6-1)</sup>:

$$
H = H_{in} + H_{prop} + H_{out}.
$$

It should be stressed that the swap-test is implemented in a non-local manner, and therefore,  $H_{prop}$  is not local. Nevertheless, each term in  $H_{prop}$  is sparse. Reminder:

<span id="page-6-0"></span>
$$
H_{prop} = \sum_{t=1}^{T} H_t,
$$
\n<sup>(1)</sup>

$$
H_t = -\frac{1}{2}|t\rangle\langle t-1| \otimes U_t - \frac{1}{2}|t-1\rangle\langle t| \otimes U_t^{\dagger} + \frac{1}{2}(|t\rangle\langle t| + |t-1\rangle\langle t-1|) \otimes I.
$$
 (2)

Indeed, it can be verified that  $H_t$  has at most 2 non-zero entries in each row, in the case that  $U_t = C - SWAP$ , regardless of the size of the swapped registers. To prove our reduction, we show the following:

<span id="page-6-1"></span><sup>&</sup>lt;sup>3</sup>Although the unary clock can be implemented, we use the construction where the clock is implemented using  $O(\log(n))$  qubits for simplicity.

- If  $x \in L_{yes}$  then there exists  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ , such that  $\langle \psi | H | \psi \rangle \leq \frac{C}{512(T+1)^5}$ .
- If  $x \in L_{no}$  then for all  $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$ ,  $\langle \psi | H | \psi \rangle \ge \frac{C}{256(T+1)^5}$ .

Completeness: In a yes instance, the two Merlins can send identical states which are accepted with probability at least c (where c is the completeness parameter), which have the form  $|\psi_1\rangle$  =  $|\psi_2\rangle = |\chi_1\rangle \otimes \ldots \otimes |\chi_r\rangle$ . Since  $C-SWAP(|+\rangle \otimes |\chi_i\rangle \otimes |\chi_i\rangle) = |+\rangle \otimes |\chi_i\rangle \otimes |\chi_i\rangle$ , the first r steps of the verification protocol (see Thm. [4\)](#page-4-4)) have no effect. Therefore,

$$
|\eta\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle^{C} \otimes U_{t} U_{t-1} \dots U_{0} (|0^{m}\rangle^{A} \otimes |\psi_{1}\rangle^{P_{1}} \otimes |\psi_{2}\rangle^{P_{2}})
$$

where C is the clock subsystem, A is the ancilla subsystem, and  $P_1$  and  $P_2$  are the first and second proof subsystems.

Using Theorem [4,](#page-4-4) we know that in a yes instance, the two Merlins can send identical states which are accepted with probability at least c. These states are of the form  $|\psi_1\rangle = |\psi_2\rangle = |\chi_1\rangle \otimes$  $\ldots \otimes |\chi_r\rangle$ . The verifier then does the following: he first performs a swap test on each pair  $|\chi_i\rangle \otimes |\chi_i\rangle$ (characterized by the first r unitaries  $U_1, dots, U_r$ ). This does not change the state at all. He then applies the verifying procedure only on the first proof, and ancilla. Therefore,

$$
|\eta\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle^{C} \otimes U_{t} U_{t-1} \dots U_{0} (|0^{m}\rangle^{A} \otimes |\psi_{1}\rangle^{P_{1}} \otimes |\psi_{2}\rangle^{P_{2}})
$$
  
= 
$$
\left(\frac{1}{\sqrt{T+1}} \sum_{t=r+1}^{T} |t\rangle^{C} \otimes U_{t} U_{t-1} \dots U_{r+1} (|0^{m}\rangle^{A} \otimes |\psi_{1}\rangle^{P_{1}})\right) \otimes |\psi_{2}\rangle^{P_{2}}.
$$

This shows that  $|\eta\rangle$  is a tensor product state with respect to the spaces  $(C \otimes A \otimes P_1)$  on one end and  $P_2$  on the other. Kitaev's proof (see [\[KSV02,](#page-13-0) Sec. 14.4.3]) shows that

$$
\langle \eta | H | \eta \rangle \le \frac{1 - c}{T + 1},\tag{3}
$$

and by substituting  $c$ , we get,

$$
\langle \eta | H | \eta \rangle \le \frac{C}{512(T+1)^5}.\tag{4}
$$

Soundness: We first outline the three steps of the proof qualitatively. We assume that there exists a low-energy state  $|\omega\rangle = |\omega_1\rangle \otimes |\omega_2\rangle$ , and we show that:

(i) If  $|\omega\rangle$  has low energy, then  $|\omega\rangle$  is close to a history state  $|\eta\rangle$ , i.e. a state of the form  $\frac{1}{\sqrt{T}}$  $\frac{1}{T+1}\sum_{t=0}^{T} |t\rangle \otimes U_t U_{t-1} \dots U_0 |0^m\rangle \otimes |\psi\rangle$  for some  $|\psi\rangle$ . We usually write such a history state  $|\eta_{\psi}\rangle$  to mark the dependence in  $|\psi\rangle$ .

(ii) If a history state  $|\eta_{\psi}\rangle$  is close to a product state (and by (i), it is), then, the associated state  $|\psi\rangle$  is close to a tensor product state.

(iii) If  $|\psi\rangle$  is close to a tensor product state, the originating history state  $|\eta_{\psi}\rangle$  must have high energy which will contradict (i).

<span id="page-7-0"></span>**Lemma 15** (Step one). If  $\langle \omega | H | \omega \rangle \le \alpha \frac{C}{(T+1)^3}$ , for some universal constant C, then, there exists a history state  $|\eta\rangle$  s.t.  $|\langle \omega|\eta\rangle|^2 \geq 1 - \alpha$ .

*Proof.* Let  $V_{hist}$  the subspace spanned by all history states. We can verify that  $V_{hist}$  is the kernel of  $H_{init} + H_{prop}$ . We use the following Claim, which is proved in Appendix [A.](#page-14-0)

<span id="page-8-0"></span>**Claim 16.**  $\Delta(H_{init} + H_{prop}) \ge \frac{C}{(T+1)^3}$ , for some universal constant C, where  $\Delta(A)$  is the smallest non-zero eigenvalue of A.

We can write 
$$
|\omega\rangle = \sqrt{1-p}|\eta\rangle + \sqrt{p}|\eta^{\perp}\rangle
$$
, for  $|\eta\rangle \in \mathcal{V}_{hist}$ , and  $|\eta^{\perp}\rangle \in \mathcal{V}_{hist}^{\perp}$ . By assumption

$$
\alpha \frac{C}{(T+1)^3} \ge \langle \omega | H | \omega \rangle \ge \langle \omega | H_{init} + H_{prop} | \omega \rangle = p \langle \eta^{\perp} | H_{init} + H_{prop} | \eta^{\perp} \rangle \ge p \frac{C}{(T+1)^3},
$$

where the first inequality follows from the assumption of the lemma, the second uses the fact that  $H_{out} \succeq 0$  and the last inequality uses Claim [16.](#page-8-0) To conclude,  $p \le \alpha$  which implies that  $|\langle \omega | \eta \rangle|^2 = 1 - p \ge 1 - \alpha$ , as needed.  $|\langle \omega | \eta \rangle|^2 = 1 - p \ge 1 - \alpha$ , as needed.

<span id="page-8-1"></span>**Lemma 17** (Step two). Let  $|\eta_{\psi}\rangle$  a history state and  $|\psi\rangle$  such that

$$
|\eta_{\psi}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |t\rangle^{C} \otimes U_{t} U_{t-1} \dots U_{0} (|0^{m}\rangle^{A} \otimes |\psi\rangle^{P_{1}, P_{2}})
$$
  
= 
$$
\frac{1}{\sqrt{T+1}} |0\rangle^{C} |0^{m}\rangle^{A} |\psi\rangle^{P_{1}, P_{2}} + \frac{1}{\sqrt{T+1}} \sum_{t=1}^{T} |t\rangle^{C} \otimes U_{t} U_{t-1} \dots U_{1} (|0^{m}\rangle^{A} |\psi\rangle^{P_{1}, P_{2}})
$$

where we consider the following subsystems:  $C$  is the clock subsystem,  $A$  is the ancilla subsystem, and  $P_1$  and  $P_2$  are the first and second proof subsystems. If there exist two states  $|\psi_1\rangle \in C \otimes A \otimes P_1$ and  $|\psi_2\rangle \in \mathcal{P}_2$  such that  $|\langle \eta_{\psi}|(|\psi_1\rangle \otimes |\psi_2\rangle)|^2 \geq 1 - \varepsilon$  then there exists a state  $|L\rangle \in \mathcal{P}_1$  such that  $|\langle \psi | ( |L \rangle \otimes | \psi_2 \rangle)|^2 \geq 1 - \varepsilon (T + 1)$ 

Proof. We write

$$
|\psi_1\rangle = \sqrt{\alpha}|0\rangle^C|0^m\rangle^A|L\rangle^{P_1} + \sqrt{1-\alpha}\sum_{i,j:(i,j)\neq(0,0^m)}\beta_{i,j}|i\rangle^C|j\rangle^A|\psi_{i,j}\rangle^{P_1}
$$

with  $\sum_{i,j:(i,j)\neq(0,0^m)}|\beta i,j|^2=1$ . From this, we immediately have

$$
|\langle \eta_{\psi} | (|\psi_1 \rangle \otimes |\psi_2 \rangle)| \le \sqrt{\frac{\alpha}{T+1}} \cdot |\langle \psi | (|L \rangle \otimes |\psi_2 \rangle)| + \sqrt{\frac{T(1-\alpha)}{T+1}} \cdot 1
$$
  
=  $\sqrt{\alpha} \sqrt{\frac{1}{T+1}} \cdot |\langle \psi | (|L \rangle \otimes |\psi_2 \rangle)| + \sqrt{1-\alpha} \sqrt{\frac{T}{T+1}}$   
 $\le \sqrt{\frac{|\langle \psi | (|L \rangle \otimes |\psi_2 \rangle)|^2}{T+1}} + \frac{T}{T+1},$ 

where used Cauchy Schwarz in both inequalities. Therefore,

$$
|\langle \psi | (|L\rangle \otimes |\psi_2 \rangle)|^2 \ge (T+1) \cdot \left( |\langle \eta_{\psi} | (|\psi_1 \rangle \otimes |\psi_2 \rangle)|^2 - \frac{T}{T+1} \right).
$$

By using  $|\langle \eta_{\psi} | (|\psi_1 \rangle \otimes |\psi_2 \rangle)|^2 \ge 1 - \varepsilon$ , we can further bound

$$
|\langle \psi | ( |L \rangle \otimes | \psi_2 \rangle)|^2 \ge 1 - \varepsilon (T + 1).
$$

 $\Box$ 

<span id="page-9-1"></span>**Lemma 18** (Step three). Consider a history state  $|\eta_{\psi}\rangle$  with an associated state  $|\psi\rangle$ . In a no instance with soundness parameter s, if  $|\langle \psi | (|\psi_1 \rangle \otimes |\psi_2 \rangle)|^2 \geq 1 - \varepsilon$ , then  $\langle \eta_\psi | H | \eta_\psi \rangle \geq \frac{1}{T+1} (1 - s - 2\sqrt{\varepsilon}).$ 

To prove this Lemma, we will need the following Claim.

<span id="page-9-0"></span>Claim 19. Let  $\Pi$  be a projector, and  $|v_1\rangle, |v_2\rangle$  be arbitrary, and let  $q_i = \langle v_i | \Pi | v_i \rangle$ . If  $|\langle v_1 | v_2 \rangle|^2 \ge$ 1 –  $\delta$ , then,  $|q_1 - q_2| \leq \sqrt{\delta}$ .

*Proof.* To prove this claim, we use the trace distance between  $|v_1\rangle\langle v_1|$  and  $|v_2\rangle\langle v_2|$ . The trace distance is denoted  $D(|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|)$  and is equal to  $\frac{1}{2}|||v_1\rangle\langle v_1| - |v_2\rangle\langle v_2||||_1$ . We know that from the characterization of the trace distance that can be found in [\[NC00\]](#page-13-12), we have

$$
D(|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|) \ge |q_1 - q_2|.
$$

Moreover, we know by a Fuchs- van de Graaf inequality that  $D(|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|) \leq \sqrt{1 - |\langle v_1|v_2\rangle|^2}$ ([\[FvdG99\]](#page-13-14)). By putting everything together, we have

$$
|q_1 - q_2| \le D(|v_1\rangle\langle v_1|, |v_2\rangle\langle v_2|) \le \sqrt{1 - |\langle v_1|v_2\rangle|^2} \le \sqrt{\delta}.
$$

We can now prove the Lemma.

*Proof.* For every state  $|\psi\rangle$ ,

$$
\langle \eta_{\psi}|H|\eta_{\psi}\rangle = \frac{1}{T+1} \langle 0^m| \otimes \langle \psi|U_1^{\dagger} \dots U_t^{\dagger} \Pi_{reject} U_t \dots U_1 | 0^m \rangle \otimes |\psi\rangle
$$

By using Claim [19,](#page-9-0) we get

$$
\langle \eta_{\psi} | H | \eta_{\psi} \rangle \ge \frac{1}{T+1} (\langle 0^m | \otimes \langle \psi_1 | \otimes \langle \psi_2 | U_1^{\dagger} \dots U_t^{\dagger} \Pi_{reject} U_t \dots U_1 | 0^m \rangle \otimes |\psi_1 \rangle \otimes |\psi_2 \rangle - 2\sqrt{\varepsilon})
$$
  
= 
$$
\frac{1}{T+1} (\Pr(\mathcal{A} \text{ rejects} |\psi_1 \rangle \otimes |\psi_2 \rangle) - 2\sqrt{\varepsilon}
$$
  

$$
\ge \frac{1}{T+1} (1 - s - 2\sqrt{\varepsilon}),
$$

where in the last inequality, we used the fact that this is a NO instance, and therefore all tensor product states are rejected with probability at least  $1 - s$ . □

We can now combine the three steps, and prove the soundness property. Assume, by contradiction, that there exists a state  $|\omega\rangle = |\omega_1\rangle \otimes |\omega_2\rangle$ , with energy below the promise, i.e.  $\langle \omega|H|\omega\rangle \leq$  $\frac{C}{256(T+1)^5}$ .

By Lemma [15,](#page-7-0) there exists a state  $|\eta_{\psi}\rangle$  such that

<span id="page-9-2"></span>
$$
|\langle \eta_{\psi}|\omega \rangle|^2 \ge 1 - \frac{1}{256(T+1)^2}.\tag{5}
$$

Using Lemma [17,](#page-8-1) there exists  $|\phi\rangle = |\phi_1\rangle \otimes |\phi_1\rangle$  such that  $|\langle \psi | \phi \rangle|^2 \ge 1 - \frac{1}{256(T+1)}$ . By Lemma [18,](#page-9-1)

$$
\langle \eta_{\psi} | H | \eta_{\psi} \rangle \ge \frac{1}{T+1} \left( 1 - \frac{1}{T+1} - 2\sqrt{\frac{1}{256(T+1)}} \right).
$$

 $\Box$ 

Our goal is to lower bound  $\langle \omega | H | \omega \rangle$ . We have,  $| \omega \rangle = \sqrt{1 - p} | \eta_{\psi} \rangle + \sqrt{p} | \eta_{\psi}^{\perp} \rangle$ , for some  $| \eta_{\psi}^{\perp} \rangle$  that satisfies  $\langle \eta_{\psi} | \eta_{\psi}^{\perp} \rangle = 0$ , where  $0 \le p \le \frac{1}{256(T+1)^2}$  by Eq. [\(5\)](#page-9-2), therefore

$$
\langle \omega | H | \omega \rangle = (1 - p) \langle \eta_{\omega} | H | \eta_{\omega} \rangle + p \langle \eta_{\omega}^{\perp} | H | \eta_{\omega}^{\perp} \rangle + 2 \sqrt{(1 - p)p} \, Re(\langle \eta_{\omega}^{\perp} | H | \eta_{\omega} \rangle) \tag{6}
$$

We define  $\delta \equiv \frac{1}{T+1} \left(1 - \frac{1}{T+1} - 2\sqrt{\frac{1}{256(T+1)}}\right)$ . Since,  $H \succeq 0$ , clearly  $\langle \eta_\omega^{\perp} | H | \eta_\omega^{\perp} \rangle \geq 0$ . Also,  $H =$  $H_{in} + H_{prop} + H_{out}$ , and  $(H_{in} + H_{prop})|\eta_{\psi}\rangle = 0$ , and  $I \succeq H_{out} \succeq 0$  which implies  $Re(\langle \eta_{\omega}^{\perp} | H | \eta_{\omega}\rangle) =$  $Re(\langle \eta_{\omega^{\perp}} | H_{out} | \eta_{\omega} \rangle) \geq -1$ . Together, this gives

$$
\langle \omega | H | \omega \rangle \ge (1 - p)\delta - 2\sqrt{(1 - p)p}.
$$

We can lower bound the first term by using  $\delta \geq \frac{1}{2(T+1)}$  and  $p \leq \frac{1}{2}$  $\frac{1}{2}$ , and the second term by using  $\sqrt{(1-p)p} \leq \sqrt{p}$ , hence

$$
\langle \omega | H | \omega \rangle \ge \frac{1}{4(T+1)} - 2\sqrt{\frac{1}{256(T+1)^2}} = \frac{1}{8(T+1)}.
$$

This contradicts our assumption that  $\langle \omega | H | \omega \rangle \leq \frac{C}{256(T+1)^5}$ , and proves the soundness property.

To conclude, when  $c = 1 - \frac{C}{512(T+1)^4}$  and  $s = \frac{1}{T+1}$ , we showed that in a YES case, there exists a tensor product state with energy at most  $\frac{C}{512(T+1)^5}$ , and in a NO case, all tensor product states have energy at least  $\frac{C}{256(T+1)^5}$ , which completes the proof of Thm. [1.](#page-2-1)

## <span id="page-10-0"></span>4 Proof that separable k-local hamiltonian is QMA-complete

In this Section, we show that the promise problem SEPARABLE k-LOCAL HAMILTONIAN is QMA-complete. Let  $H_1, \ldots, H_m$  an instance of SEPARABLE k-LOCAL HAMILTONIAN. We partition the workspace of qubits into disjoints sets A and B, each corresponding to  $n/2$  qubits. Let  $A_i \subset A$  (resp.  $B_i \subset B$ ) the space of qubits in A (resp. B) on which  $H_i$  acts.  $H_i$  acts on k qubits represented by the space  $A_i \otimes B_i$ . We can have  $A_i = \emptyset$  or  $B_i = \emptyset$ . Let  $H = \sum_i H_i$  (where the summation is over the extension of the  $H_i$ 's to the entire Hilbert space). The size of the instance is  $N = n + m \cdot 2^k$ . The term  $2^k$ follows from the need of  $O(2^k)$  classical bits to describe a k-local Hamiltonian.

We use Definition [9](#page-4-5) to characterize SEPARABLE k-LOCAL HAMILTONIAN. We are in a YES instance if there exists  $\rho = \rho_A \otimes \rho_B$  such that  $tr(H\rho) \leq a$ , with  $\rho_A \in \mathcal{A}$  and  $\rho_B \in \mathcal{B}$ . We are in a NO instance if for all  $ρ = ρ<sub>A</sub> ⊗ ρ<sub>B</sub>$  with  $ρ<sub>A</sub> ∈ A$  and  $ρ<sub>B</sub> ∈ B$ , we have tr( $Hρ$ ) ≥ b

Note that this problem is QMA-hard hard. Indeed, if we consider that all the Hamiltonians  $H_i$ act only on A, which means that for all i,  $B_i = \emptyset$ , we obtain an instance of the k-local Hamiltonian problem which is QMA-complete hence SEPARABLE k-LOCAL HAMILTONIAN is QMA-hard. It remains to be shown that separable k-local hamiltonian is in QMA.

To show this, we use the fact that another problem, consistency of local density matrices (see Definition [12\)](#page-5-3), is in QMA. More precisely, we consider the consistency of local DENSITY MATRICES problem with  $\beta = \frac{b-a}{8m}$ . It was shown by Liu [\[Liu06\]](#page-13-11) that this problem is in QMA. We now describe the QMA procedure for separable k-local hamiltonian.

### QMA protocol for SEPARABLE  $k -$  LOCAL HAMILTONIAN

Let  $H_1 \ldots, H_m$  an instance of SEPARABLE  $k -$  LOCAL HAMILTONIAN. Suppose this is a YES instance. Let  $\rho = \rho_A \otimes \rho_B$  such that  $tr(H\rho) \le a$  For each Hamiltonian  $H_i$ , do the following:

- The prover sends a classical description of the state  $\rho_i = \rho^{A_i} \otimes \rho^{B_i}$  where  $\rho^{A_i} =$  $\text{Tr}_{A/A_i}(\rho_A)$  and  $\rho^{B_i} = \text{Tr}_{B/B_i}(\rho_B)$ . This requires sending  $O(m \cdot 2^k) = O(N)$  classical bits.
- The prover proves to the verifier that the reduced density matrices  $\{\rho^{A_i}\}_{i\in[1,m]}$  form a yes instance of the consistency of local density matrices problem. He also proves that the reduced density matrices  $\{\rho_{B_i}\}_{i\in[1,m]}$  form a YES instance of the CONsistency of local density matrices problem.
- Once the verifier is convinced that the reduced density matrices are consistent, he calculates the value  $E = \sum_i \text{tr}(H_i \rho_i)$  and accepts if  $E \leq a$ .

### 4.1 Proof that the protocol works

*Proof.* Completeness: Suppose we are in a yes instance. This means that there exists  $\rho = \rho_A \otimes \rho_B$ such that  $tr(H\rho) \leq a$ . The prover sends a classical description of the  $\rho_{A_i}$  and  $\rho_{B_i}$  where  $\rho_{A_i} =$  $\text{Tr}_{A/A_i}(\rho_A)$  and  $\rho_{B_i} = \text{Tr}_{B/B_i}(\rho_B)$ . Clearly, these reduced density matrices are consistent with  $\rho_A$ and  $\rho_B$  so the consistency test will pass with probability greater than 2/3. Then, we have

$$
\operatorname{tr}(H\rho) = \sum_i \operatorname{tr}(H_i\rho) = \sum_i \operatorname{tr}(H_i(\rho_{A_i} \otimes \rho_{B_i})) = E \le a.
$$

We conclude that the verifier will accept with probability at least  $\frac{2}{3}$ .

Soundness: Suppose we are in a no instance. The prover sends classical descriptions of the states  $\rho_{A_i}, \rho_{B_i}$ . We distinguish two cases:

- These reduced density matrices fail the consistency test. The verifier accepts with probability smaller than  $\frac{1}{3}$ .
- These reduced density matrices pass the consistency test with probability at least  $\frac{2}{3}$ . This means that there exist two quantum states  $\sigma_A, \sigma_B$  such that if we define  $\sigma_{A_i} = \text{Tr}_{A/A_i}(\sigma_A)$ and  $\sigma_{B_i} = \text{Tr}_{B/B_i}(\sigma_B)$ , we have :

$$
\forall i, ||\sigma_{A_i} - \rho_{A_i}||_1 \le \frac{b-a}{8m}
$$
 and  $||\sigma_{B_i} - \rho_{B_i}||_1 \le \frac{b-a}{8m}$ .

Since we are in a NO instance, for every  $\sigma_A, \sigma_B$  we have

$$
\operatorname{tr}(H(\sigma_A \otimes \sigma_B)) = \sum_i \operatorname{tr}(H_i(\sigma_{A_i} \otimes \sigma_{B_i})) \geq b.
$$

For each *i*, we have  $||(\rho_{A_i} \otimes \rho_{B_i}) - (\sigma_{A_i} \otimes \sigma_{B_i})||_1 \leq ||\rho_{A_i} - \sigma_{A_i}||_1 + ||\rho_{B_i} - \sigma_{B_i}||_1 \leq \frac{b-a}{4m}$ , where the first inequality follows from the subadditivity of the trace distance with respect to tensor products. We now use the following Claim

Claim 20 ([\[NC00\]](#page-13-12)). Let  $\rho, \sigma$  two quantum states with  $||\rho - \sigma||_1 = \delta$ . We have for any positive semidefinite matrix  $H \leq I$ ,  $|tr(H(\rho)) - tr(H(\sigma))| \leq \delta/2$ .

Since we do have  $H_i$  positive semi definite and  $H_i \leq I$  for each i, we have for all i

$$
\operatorname{tr}(H_i(\rho_{A_i} \otimes \rho_{B_i})) \geq \operatorname{tr}(H_i(\sigma_{A_i} \otimes \sigma_{B_i})) - \frac{b-a}{2m}.
$$

Putting this all together, we have

$$
E = \sum_i \text{tr}(H_i(\rho_{A_i} \otimes \rho_{B_i})) \ge \sum_i \text{tr}(H_i(\sigma_{A_i} \otimes \sigma_{B_i})) - m \cdot \frac{b-a}{2m} \ge b - \frac{a-b}{2} = \frac{a+b}{2},
$$

therefore, the verifier rejects.

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## <span id="page-14-0"></span>A Lower bounding the spectral gap

In this appendix we prove Claim [16.](#page-8-0) The proof has a similar structure to the one in [\[KSV02\]](#page-13-0). We will first need a few definitions. Given a Hilbert space  $\mathcal{H}$  and a subspace  $\mathcal{L}$ , the subspace  $\mathcal{L}^{\perp}$  is the orthogonal complement of the subspace  $\mathcal L$  (see e.g. [\[Per91\]](#page-13-15)). Given two subspaces  $\mathcal L_1,\mathcal L_2$ , the angle  $0 \leq \theta(\mathcal{L}_1, \mathcal{L}_2) \leq \frac{\pi}{2}$  $\frac{\pi}{2}$  between the subspaces is:

$$
\cos(\theta) \equiv \max_{|\psi_1\rangle \in \mathcal{L}_1, |\psi_2\rangle \in \mathcal{L}_2} |\langle \psi_1 | \psi_2 \rangle|.
$$

Given a Hamiltonian  $A \succeq 0$ , we define  $\Delta(A)$  to be the smallest non-zero eigenvalue of A. We use the notation  $A \succeq c$  as a shorthand for  $A - cI \succeq 0$ .

<span id="page-14-1"></span>**Lemma 21** ([\[KSV02,](#page-13-0) Lemma 14.4]). Let  $A_1, A_2$  be positive-semidefinite operators, and  $\mathcal{L}_1, \mathcal{L}_2$  their null subspaces respectively, where  $\mathcal{L}_1 \cap \mathcal{L}_2 = \{0\}$ . Suppose further that  $\Delta(A_1) \geq v$  and  $\Delta(A_2) \geq v$ . Then,

$$
A_1 + A_2 \succeq v(1 - \cos(\theta)),
$$

where  $\theta$  is the angle between  $\mathcal{L}_1$  and  $\mathcal{L}_2$ .

We will use a slightly different version.

<span id="page-14-2"></span>**Corollary 22.** Let  $A_1, A_2$  be positive-semidefinite operators, and  $\mathcal{L}_1, \mathcal{L}_2$  their null subspaces respectively, where  $\mathcal{L}_1 \cap \mathcal{L}_2 \equiv \mathcal{L}$ . Suppose further that  $\Delta(A_1) \geq v$  and  $\Delta(A_2) \geq v$ . Then,

$$
\Delta(A_1 + A_2) \ge v(1 - \cos(\theta)),
$$

where  $\theta$  is the angle between  $\mathcal{L}_1 \cap \mathcal{L}^{\perp}$  and  $\mathcal{L}_2 \cap \mathcal{L}^{\perp}$ .

The corollary follows from applying Lemma [21,](#page-14-1) to  $A_1$  and  $A_2$  with the domain and codomain restricted to  $\mathcal{L}^{\perp}$ .

We use Cor. [22](#page-14-2) where we substitute  $A_1 = W^{\dagger} H_{in} W$  and  $A_2 = W^{\dagger} H_{prop} W$ , where

$$
W = \sum_{t=0}^{T} |t\rangle\langle t| \otimes U_t \dots U_1.
$$

In this case, the analysis in [\[KSV02,](#page-13-0) Eq.  $(14.15),(14.16)$ ] shows that  $v \ge \frac{c'}{(L+1)^2}$ , and we will show that

<span id="page-14-3"></span>
$$
\cos^2(\theta) \le 1 - \frac{1}{T+1},\tag{7}
$$

which together gives the desired result.

It can be verified (see [\[KSV02,](#page-13-0) Eq. (14.13),(14.14)] for a full analysis) that

$$
\mathcal{L}_1 = |0\rangle^C \otimes |0^m\rangle^A \otimes \mathcal{H}^{P_1, P_2} \bigoplus_{i=1}^{2^m - 1} |i\rangle^C \otimes \mathcal{H}^{A, P_1, P_2}
$$

$$
\mathcal{L}_2 = |\alpha\rangle^C \otimes \mathcal{H}^{A, P_1, P_2},
$$

where  $|\alpha\rangle = \frac{1}{\sqrt{T}}$  $\frac{1}{T+1}\sum_{t=0}^{T} |t\rangle$ , and the superscripts denote the subsystems (see Lemma [17\)](#page-8-1). Therefore,

$$
\mathcal{L} \equiv \mathcal{L}_1 \cap \mathcal{L}_2 = |\alpha\rangle^C \otimes |0^m\rangle^A \otimes \mathcal{H}^{P_1, P_2}
$$

$$
\mathcal{L}_2 \cap \mathcal{L}^{\perp} = |\alpha\rangle^C \otimes (\bigoplus_{i=1}^{2^m-1} |i\rangle^A) \otimes \mathcal{H}^{P_1,P_2},
$$

Using these definitions we get:

$$
\cos^2(\theta) = \max_{|\psi_1\rangle \in \mathcal{L}_1 \cap \mathcal{L}^\perp, |\psi_2\rangle \in \mathcal{L}_2 \cap \mathcal{L}^\perp} |\langle \psi_1 | \psi_2 \rangle|^2
$$
\n(8)

$$
\leq \max_{|\psi_1\rangle \in \mathcal{L}_1, \ |\psi_2\rangle \in \mathcal{L}_2 \cap \mathcal{L}^\perp} |\langle \psi_1 | \psi_2 \rangle|^2 \tag{9}
$$

$$
= \max_{|\psi\rangle \in \mathcal{L}_2 \cap \mathcal{L}^\perp} \langle \psi | \Pi_{\mathcal{L}_1} | \psi \rangle, \tag{10}
$$

where  $\Pi_{\mathcal{L}_1} = |0\rangle\langle 0|^{C} \otimes |0^{m}\rangle\langle 0^{m}|^{A} \otimes I^{P} + \sum_{t=1}^{T} |t\rangle\langle t|^{C} \otimes I^{A,P_1,P_2}$  is the projection onto the space  $\mathcal{L}_1$ . Any state  $|\psi\rangle \in \mathcal{L}_2 \cap \mathcal{L}^{\perp}$  can be written in the form  $|\alpha\rangle^C \otimes |\beta\rangle^{A,P_1,P_2}$ , where  $|0^m\rangle\langle 0^m|^A \otimes I^{P_1,P_2}|\beta\rangle = 0$ , therefore we can further bound:

$$
\cos^{2}(\theta) \leq \langle \alpha | \otimes \langle \beta | | 0 \rangle \langle 0 |^{C} \otimes | 0^{m} \rangle \langle 0^{m} |^{A} \otimes I^{P_{1}, P_{2}} | \alpha \rangle \otimes | \beta \rangle + \langle \alpha | \otimes \langle \beta | \sum_{t=1}^{T} | t \rangle \langle t |^{C} \otimes I^{A, P_{1}, P_{2}} | \alpha \rangle \otimes | \beta \rangle
$$
  
=  $1 - \frac{1}{T+1}$ ,

where the equality follows from the observation that the first term is 0 (see the property of  $|\beta\rangle$ ) mentioned above), and that  $\langle \alpha | \sum_{t=1}^T |t\rangle\langle t | |\alpha \rangle = \frac{T}{T+1}$ . This gives Eq. [7,](#page-14-3) and completes the proof.